

Realizing isomorphisms between first homology groups of closed 3-manifolds by borromean surgeries

Delphine Moussard*

Abstract

We refine Matveev's result asserting that any two closed oriented 3-manifolds can be related by a sequence of borromean surgeries if and only if they have isomorphic first homology groups and linking pairings. Indeed, a borromean surgery induces a canonical isomorphism between the first homology groups of the involved 3-manifolds, which preserves the linking pairing. We prove that any such isomorphism is induced by a sequence of borromean surgeries. As an intermediate result, we prove that a given algebraic square finite presentation of the first homology group of a 3-manifold, which encodes the linking pairing, can always be obtained from a surgery presentation of the manifold.

MSC: **57M27** 57M25 57N10 57N65

Keywords: Homology groups; linking pairing; 3-manifold; surgery presentation; linking matrix; stable equivalence; borromean surgery; Y-graph; Y-link.

Contents

1 Preliminaries	3
2 Realizing isomorphisms between linking pairings	5
3 Kirby transformations	9
4 Presentations of $H_1(M; \mathbb{Z})$, topological version	11
5 Presentations of $H_1(M; \mathbb{Z})$, algebraic version	14
6 Example	18

*The author was supported by the Italian FIRB project "Geometry and topology of low-dimensional manifolds", RBFR10GHHH.

Introduction

A well-known result of Matveev [Mat87, Theorem 2] asserts that any two closed connected oriented 3-manifolds can be related by a sequence of borromean surgeries if and only if they have isomorphic first homology groups and linking pairings. It is an important result, which is useful in particular in the Goussarov-Habiro theory of finite type invariants of 3-manifolds, based on borromean surgeries. The more direct application of Matveev's result in this theory is the fact that the degree 0 invariants are exactly encoded by the isomorphism class of the first homology group equipped with the linking pairing.

One can study 3-manifolds equipped with an additional structure, as a homological parametrization (to study the Reidemeister torsion, see [Mas10]) or a fixed cohomology class (to define Costantino-Geer-Patureau's invariants [CGP14]). In these cases, a theory of finite type invariants similar to the Goussarov-Habiro's one can be defined. To study such theories, it would be useful to have a refined version of Matveev's result. It is the goal of this article to state and prove such a refinement. More precisely, a borromean surgery induces a canonical isomorphism between the first homology groups of the involved 3-manifolds, which preserves the linking pairing. We prove that any such isomorphism is induced by a sequence of borromean surgeries.

Historical overview: Borromean surgeries were introduced by Matveev in [Mat87]. The formalism used in this article is due to Goussarov [Gou99]. An exposition of Matveev's and Goussarov's definitions, and of the relation between them, is given in [Mas03, §1]. An equivalent move has been defined by Habiro [Hab00].

Let us recall the main lines of the proof of the mentioned result of Matveev. The fact that a borromean surgery preserves the isomorphism class of the first homology group equipped with the linking pairing is easy, and we focus on the converse. Consider two closed connected oriented 3-manifolds with isomorphic first homology groups and linking pairings, and present them by surgery links in S^3 . When the associated linking matrices are stably equivalent, *i.e.* are related by stabilizations/destabilizations and unimodular congruences, Matveev proves by topological manipulations that the two surgery links are related by borromean surgeries [Mat87, Lemma 2]. This can also be achieved by applying results of Murakami-Nakanishi [MN89] and Garoufalidis-Goussarov-Polyak [GGP01] (see Section 2, Proposition 2.3). To conclude, one needs to prove that any two matrices which define isomorphic groups and pairings are stably equivalent. This was first proved by Kneser and Puppe [KP53, Satz 3] in the case of a finite group of odd order, and then by Durfee [Dur77, Corollary 4.2] for any finite group, with a different and more direct method. The case of an infinite group deduces easily (see Kyle [Kyl54, Lemma 1]).

In [Mas03], Massuyeau gives the analogue of Matveev's result in the case of 3-manifolds

with spin structure, proving that two such manifolds are related by spin borromean surgeries if and only if they have isomorphic first homology groups and linking pairings, and equal Rochlin invariants modulo 8.

Plan of the paper: In the first section, we recall the definitions of the linking pairing and of borromean surgeries. In Section 2, we state the main theorem, and we reduce its proof to another result, namely the fact that a given square finite presentation of the first homology group of a closed connected oriented 3-manifold, which encodes the linking pairing, can always be obtained from a surgery presentation of the manifold. The next three sections are devoted to the proof of this latter theorem. We first review Kirby calculus in Section 3, and we give two different proofs of the second theorem in Sections 4 and 5. The first proof is of topological nature and quite direct. The second proof is of algebraic nature; it is more technical, but also more constructive. We end with an example in Section 6.

Convention: Unless otherwise mentioned, a 3-manifold is closed, connected and oriented. The homology class of a curve γ in a manifold is denoted by $[\gamma]$. If $L \subset S^3$ is a link, $S^3(L)$ is the manifold obtained by surgery on L . If K is a knot in a 3-manifold, a *meridian* of K is the boundary of an embedded disk which meets K exactly once. If Σ and γ are respectively a surface and a curve embedded in a 3-manifold, which intersect transversely, then $\langle \Sigma, \gamma \rangle$ denotes their algebraic intersection number.

Acknowledgments: I wish to thank Christine Lescop for interesting suggestion.

1 Preliminaries

Linking pairings. Let M be a 3-manifold. If K and J are knots in M whose homology classes in $H_1(M; \mathbb{Z})$ have finite orders, one can define the linking number of K and J as follows. Let d be the order of $[K] \in H_1(M; \mathbb{Z})$, and let $T(K)$ be a tubular neighborhood of K . A *Seifert surface* of K is a surface Σ embedded in $M \setminus \text{Int}(T(K))$ whose boundary $\partial \Sigma \subset \partial T(K)$ satisfies $[\partial \Sigma] = d[K]$ in $H_1(T(K); \mathbb{Z})$. Such a surface always exists and can be chosen transverse to J . Define the linking number of K and J as $lk(K, J) = \frac{1}{d} \langle \Sigma, J \rangle$. It does not depend on the choice of the surface Σ , and it is symmetric. If the knots K and J are modified within their respective homology classes, the value of the linking number is modified by an integer. This allows the following definition. The *linking pairing* of M

is the \mathbb{Z} -bilinear form

$$\begin{aligned} \varphi_M : \quad \text{Tor}(H_1(M; \mathbb{Z})) \times \text{Tor}(H_1(M; \mathbb{Z})) &\rightarrow \mathbb{Q}/\mathbb{Z} \\ ([K], [J]) &\mapsto lk(K, J) \bmod \mathbb{Z} \end{aligned}$$

where Tor stands for the torsion subgroup. This form is symmetric and non-degenerate.

Framings. A 1-dimensional object is *framed* if it is equipped with a fixed normal vector field. In the case of a knot K in a 3-manifold, it is equivalent to fix a *parallel* of the knot, *i.e.* a simple closed curve $\ell(K)$ on the boundary of a tubular neighborhood $T(K)$ of K which is isotopic to K in $T(K)$. If K is a framed knot in a 3-manifold, with fixed parallel $\ell(K)$, whose homology class has finite order, define the self-linking of K by $lk(K, K) = lk(K, \ell(K))$.

Borromean surgeries. The *standard Y-graph* is the graph $\Gamma_0 \subset \mathbb{R}^2$ represented in Figure 1. With Γ_0 is associated a regular neighborhood $\Sigma(\Gamma_0)$ of Γ_0 in the plane. The

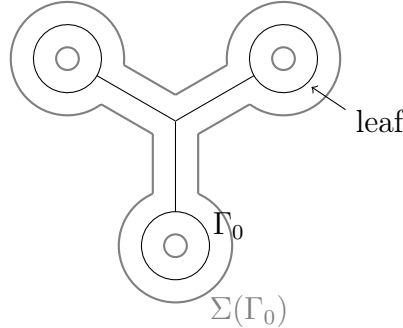


Figure 1: The standard Y-graph

surface $\Sigma(\Gamma_0)$ is oriented with the usual convention. Let M be a 3-manifold and let $h : \Sigma(\Gamma_0) \rightarrow M$ be an embedding. The image Γ of Γ_0 is a *Y-graph*, endowed with its *associated surface* $\Sigma(\Gamma) = h(\Sigma(\Gamma_0))$. The looped edges of Γ are its *leaves*. The Y-graph Γ is equipped with the framing induced by $\Sigma(\Gamma)$.

Let Γ be a Y-graph in a 3-manifold M (which may have a non-empty boundary). Let $\Sigma(\Gamma)$ be its associated surface. In $\Sigma(\Gamma) \times [-1, 1]$, associate with Γ the six-component link L represented in Figure 2, with the blackboard framing. The *borromean surgery* on Γ is the usual surgery along the framed link L . The manifold obtained from M by surgery on Γ is denoted by $M(\Gamma)$.

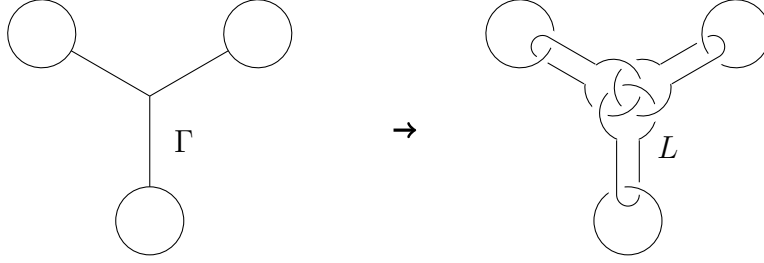


Figure 2: Y-graph and associated surgery link

The borromean surgery on a Y-graph Γ in a 3-manifold M can be realized by removing the interior of a tubular neighborhood N of Γ and gluing instead another genus 3 handlebody, via an isomorphism of their boundaries which is the identity in homology ([Mat87, Section 6], see also [Mas03, Lemma 1]). This implies that $\ker(H_1(\partial N; \mathbb{Z}) \rightarrow H_1(N; \mathbb{Z})) = \ker(H_1(\partial N; \mathbb{Z}) \rightarrow H_1(N(\Gamma); \mathbb{Z})) \subset H_1(\partial N; \mathbb{Z})$, where we consider the applications induced in homology by the natural inclusions.

A *Y-link* in a 3-manifold is a disjoint union of Y-graphs. The *borromean surgery on a Y-link* is given by the simultaneous borromean surgeries on all its components. Note that a finite sequence of borromean surgeries can always be performed by a single borromean surgery on a Y-link.

2 Realizing isomorphisms between linking pairings

We begin with the lemma whose converse is the object of the article.

Lemma 2.1. *Let M be a 3-manifold. Let Γ be a Y-link in M . The surgery on Γ induces a canonical isomorphism $\xi_\Gamma : H_1(M; \mathbb{Z}) \xrightarrow{\cong} H_1(M(\Gamma); \mathbb{Z})$ which preserves the linking pairing.*

Proof. Let N be a tubular neighborhood of Γ in M , and set $X = M \setminus N$. The Mayer-Vietoris sequence associated with $M = X \cup N$ yields the exact sequence:

$$H_1(\partial N; \mathbb{Z}) \rightarrow H_1(N; \mathbb{Z}) \oplus H_1(X; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z}) \rightarrow 0.$$

Since $H_1(\partial N; \mathbb{Z}) \cong H \oplus H_1(N; \mathbb{Z})$, where $H = \ker(H_1(\partial N; \mathbb{Z}) \rightarrow H_1(N; \mathbb{Z}))$, we have $H_1(M; \mathbb{Z}) \cong \frac{H_1(X; \mathbb{Z})}{H}$. We have seen in the previous section that $H = \ker(H_1(\partial N; \mathbb{Z}) \rightarrow$

$H_1(N(\Gamma); \mathbb{Z}) \subset H_1(\partial N; \mathbb{Z})$, hence we have similarly $H_1(M(\Gamma); \mathbb{Z}) \cong \frac{H_1(X; \mathbb{Z})}{H}$, and it follows that $H_1(M; \mathbb{Z})$ and $H_1(M(\Gamma); \mathbb{Z})$ are canonically identified.

The canonical isomorphism $\xi_\Gamma : H_1(M; \mathbb{Z}) \rightarrow H_1(M(\Gamma); \mathbb{Z})$ can be described as follows. Let $\eta \in H_1(M; \mathbb{Z})$. Represent η by a knot K in M disjoint from Γ . The knot K is not affected by the surgery. The image $\xi_\Gamma(\eta)$ is the homology class of $K \subset M(\Gamma)$.

Let us check that the linking numbers are preserved. Let K and J be knots in M , disjoint from Γ , whose homology classes have finite orders. Let Σ be a Seifert surface of K , transverse to Γ and J . We may assume that the only edges of Γ that Σ meets are its leaves. The surgery modifies the tubular neighborhood N of Γ . At each point of $\Sigma \cap \Gamma$, remove a little disk from Σ and replace it, after surgery, with the surface drawn in Figure 3, where the apparent boundary inside $N(\Gamma)$ bounds a disk in the corresponding reglued torus. This provides a surface Σ' in $M(\Gamma)$, which is again a Seifert surface of K , and such

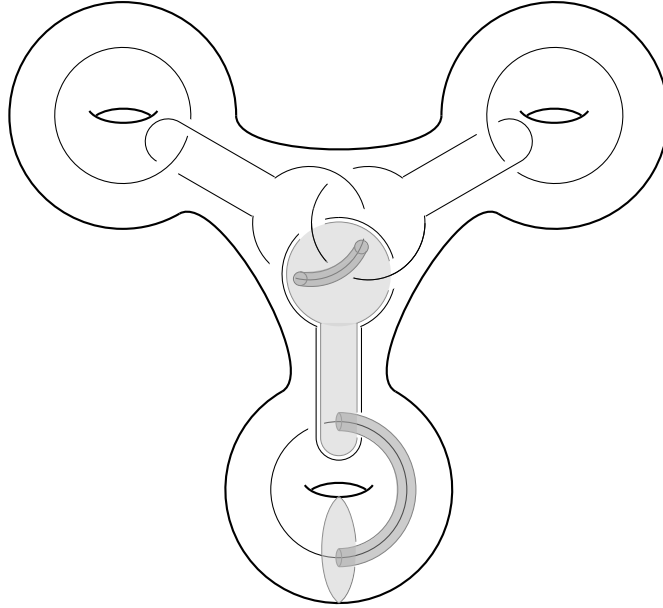


Figure 3: Surface in the reglued handlebody

that the algebraic intersection numbers $\langle \Sigma, J \rangle$ and $\langle \Sigma', J \rangle$ are equal. □

The main result of the article is the following converse of Lemma 2.1.

Theorem 2.2. *Let M and N be 3-manifolds. Let $\xi : H_1(M; \mathbb{Z}) \xrightarrow{\cong} H_1(N; \mathbb{Z})$ be an isomorphism which preserves the linking pairing. Then there is a Y -link Γ in M such that $N \cong M(\Gamma)$ and $\xi = \xi_\Gamma$.*

The proof of Theorem 2.2 uses surgery presentations of the manifolds. In order to have an associated presentation of the first homology group, we need to fix an orientation of the surgery link.

An *oriented surgery presentation* of a 3-manifold M is an oriented framed link $L \subset S^3$ such that M is obtained from S^3 by surgery on L , with a given numbering $L = \sqcup_{1 \leq i \leq n} L_i$ of the knot components of L . The *linking matrix* of L is the matrix $A(L)$ defined by $(A(L))_{ij} = -lk(L_i, L_j)$ (this convention with a minus sign is unusual, but simplifies the expression of the linking pairing in terms of the linking matrix). For $1 \leq i \leq n$, let m_i be an *oriented meridian* of L_i , i.e. such that $lk(L_i, m_i) = 1$. The *presentation* of $H_1(M; \mathbb{Z})$ induced by the oriented surgery presentation L is given by the family of generators $\mathbf{m} = ([m_1], \dots, [m_n])$ up to the relations given by the columns of $A(L)$. It is well known that any 3-manifold admits an (oriented) surgery presentation (Lickorish [Lic62], Wallace [Wal60]).

Proposition 2.3. *Let M (resp. M') be a 3-manifold with oriented surgery presentation L (resp. L'). Denote by \mathbf{m} (resp. \mathbf{m}') the associated family of generators of $H_1(M; \mathbb{Z})$ (resp. $H_1(M'; \mathbb{Z})$). If $A(L) = A(L')$, then there is a Y -link Γ in M such that $M' \cong M(\Gamma)$ and $\xi_\Gamma(\mathbf{m}) = \mathbf{m}'$.*

Proof. A Δ -move on an oriented framed link is a local move as represented in Figure 4 involving three components of the link (with possible repetitions), which keeps unchanged the orientations and the framings. By [MN89, Theorem 1.1], any two oriented framed

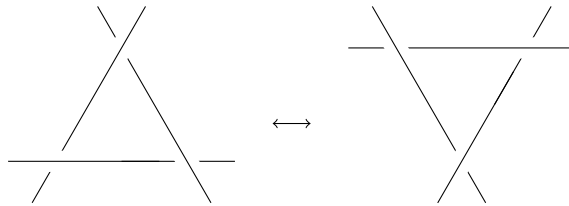


Figure 4: Δ -move

links with the same linking matrix are related by a sequence of Δ -moves. By [GGP01, Lemma 2.1], such a Δ -move can be realised by a borromean surgery (see Figure 5), which

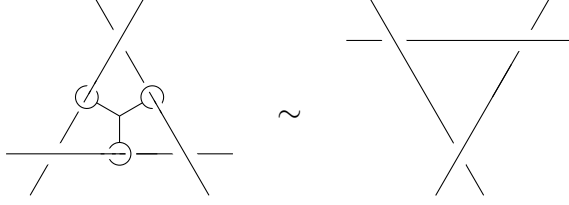


Figure 5: Borromean surgery realizing a Δ -move

means that the pair (3-manifold, link) obtained by the surgery is homeomorphic to the pair (S^3, link) obtained by the Δ -move.

Finally, there exists a Y-link Γ in $S^3 \setminus L$ such that $S^3(\Gamma) \cong S^3$, and the copy of L in $S^3(\Gamma)$ is isotopic to L' . Since the surgery on Γ is performed in the complement of L , we can consider the copy of Γ in M , and we have $M(\Gamma) \cong M'$. The assertion on ξ_Γ follows from the fact that the meridians of the components of L are not affected by the Δ -moves. \square

We shall prove that we can choose a surgery presentation providing a fixed presentation of the first homology group. Let us fix a few algebraic definitions.

A *linking matrix* is a square symmetric matrix with integral coefficients. A linking matrix A is *non-degenerate* if $\det(A) \neq 0$, and *admissible* if $A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$ where A_0 is a non-degenerate linking matrix. By [Kyl54, Lemma 1], any linking matrix is congruent over the integers to an admissible linking matrix.

Let A be a linking matrix. If $B = \begin{pmatrix} D & 0 \\ 0 & A \end{pmatrix}$, where D is a diagonal matrix whose diagonal terms are ± 1 , the matrix B is a *stabilization* of A and A is a *destabilization* of B .

A *linking pairing* on a finite abelian group H is a symmetric bilinear pairing $\varphi : H \times H \rightarrow \mathbb{Q}/\mathbb{Z}$. A *linked group* is a finitely generated abelian group H whose torsion subgroup $\text{Tor}(H)$ is equipped with a linking pairing. Let A be a linking matrix of size n and rank r . The matrix A can be written $A = {}^tP \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} P$, where A_0 is a non-degenerate linking matrix, P is an integral matrix invertible over the integers, and tP is the transpose of P . An *A-presentation* of a linked group H is a family $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n) \subset H$ such that:

- the following sequence is exact, where the first map is given by A in the canonical

bases and the second map sends the canonical basis on $\boldsymbol{\eta}$,

$$\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \longrightarrow H \longrightarrow 0$$

- $\varphi((P^{-1}\eta)_i, (P^{-1}\eta)_j) = (A_0^{-1})_{ij} \bmod \mathbb{Z}$ for $1 \leq i, j \leq r$, where φ is the linking pairing on $\text{Tor}(H)$.

Let M be a 3-manifold. The group $H_1(M; \mathbb{Z})$ has a natural structure of linked group given by the linking pairing φ_M of M . If L is an oriented surgery presentation of M , the presentation of $H_1(M; \mathbb{Z})$ induced by L is an $A(L)$ -presentation.

Theorem 2.4. *Let M be a 3-manifold. Let A be a linking matrix. Assume there is an A -presentation $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)$ of $H_1(M; \mathbb{Z})$. Then there is an oriented surgery presentation L of M such that $A(L)$ is a stabilization of A , and L induces the $A(L)$ -presentation $\boldsymbol{\eta}' = (0, \dots, 0, \eta_1, \dots, \eta_n)$.*

This result is the purpose of the next three sections. We first review Kirby calculus in Section 3, and we give two independant proofs of the theorem in Sections 4 and 5.

Proof of Theorem 2.2. Let L_1 be an oriented surgery presentation of M . Let $\boldsymbol{\eta}$ be the associated $A(L_1)$ -presentation of $H_1(M; \mathbb{Z})$. The family $\boldsymbol{\nu} = (\xi(\eta_1), \dots, \xi(\eta_n))$ is an $A(L_1)$ -presentation of $H_1(N; \mathbb{Z})$. By Theorem 2.4, there is an oriented surgery presentation L_2 of N such that $A(L_2)$ is a stabilization of $A(L_1)$, and L_2 induces the $A(L_2)$ -presentation $\boldsymbol{\nu}' = (0, \dots, 0, \xi(\eta_1), \dots, \xi(\eta_n))$. Define a surgery link L'_1 by adding (± 1) -framed trivial components to L_1 , and renumber the components, so that $A(L'_1) = A(L_2)$ and L'_1 induces the $A(L_2)$ -presentation $\boldsymbol{\eta}' = (0, \dots, 0, \eta_1, \dots, \eta_n)$. Conclude with Proposition 2.3. \square

3 Kirby transformations

By a well-known theorem of Kirby [Kir78], any two oriented surgery presentations of a 3-manifold M are related by a finite sequence of the following K1 and K2 moves.

- K1 (stabilization/destabilization): add or remove a (± 1) -framed trivial component, unlinked with the other components, to the surgery link.
- K2: add or subtract a component of the surgery link to another (see Figure 6).



Figure 6: K2 move

The effect of a Kirby move on the linking matrix is a stabilization/destabilization in the case of a K1 move, and a unimodular congruence in the case of a K2 move (where *unimodular* means that the congruence matrix has determinant 1). Conversely, a stabilization or a unimodular congruence on the linking matrix can be realized by a finite sequence of Kirby moves. Note that this converse does not hold for a destabilization. In order to prove Theorem 2.4, we need to understand the effect of a Kirby move, performed on a surgery link L , on the associated $A(L)$ -presentation of the first homology group. The case of a K1 move is obvious: we add or remove a trivial generator to the presentation. Let us consider the case of a K2 move.

Let M be a 3-manifold with oriented surgery presentation $L = \sqcup_{1 \leq k \leq n} L_k$. Define another oriented surgery presentation $L' = \sqcup_{1 \leq k \leq n} L'_k$ of M by adding the component L_j to L_i (Fig. 6). The reason why L' is again a surgery presentation of M is the following. We can perform first the surgery on the component L_j and then switch the component L_i along the meridian disk bounded by the parallel of L_j . Now the effect on the meridians of the components of L is represented in Figure 7. If m_k (resp. m'_k) is an oriented meridian of L_k (resp. L'_k), then $[m'_k] = [m_k]$ for $k \neq j$ and $[m'_j] = [m_j] - [m_i]$. The induced congruence between the linking matrices is $A(L') = T_{ij}A(L)T_{ji}$, where $T_{k\ell}$ is the transvection matrix defined as $T_{k\ell} = I + E_{k\ell}$ and $E_{k\ell}$ is the matrix whose only non-trivial coefficient is a 1 at the k -th row and ℓ -th column. If \mathbf{m} (resp. \mathbf{m}') is the presentation of $H_1(M; \mathbb{Z})$ associated with L (resp. L'), we have $T_{ji}\mathbf{m}' = \mathbf{m}$. If the component L_j is subtracted to L_i , then the transvection matrix which appears is $T_{ji}^{-1} = I - E_{ji}$. Since any *unimodular matrix* (square integral matrix of determinant 1) is a product of transvection matrices, we obtain the following statement.

Lemma 3.1. *Let M be a 3-manifold. Let L be an oriented surgery presentation of M . Let \mathbf{m} be the associated presentation of $H_1(M; \mathbb{Z})$. Consider $A = {}^tPA(L)P$, where P is a unimodular matrix. Then there is an oriented surgery presentation L' of M , that can be obtained from L by a sequence of K2 moves, such that $A(L') = A$ and the associated*

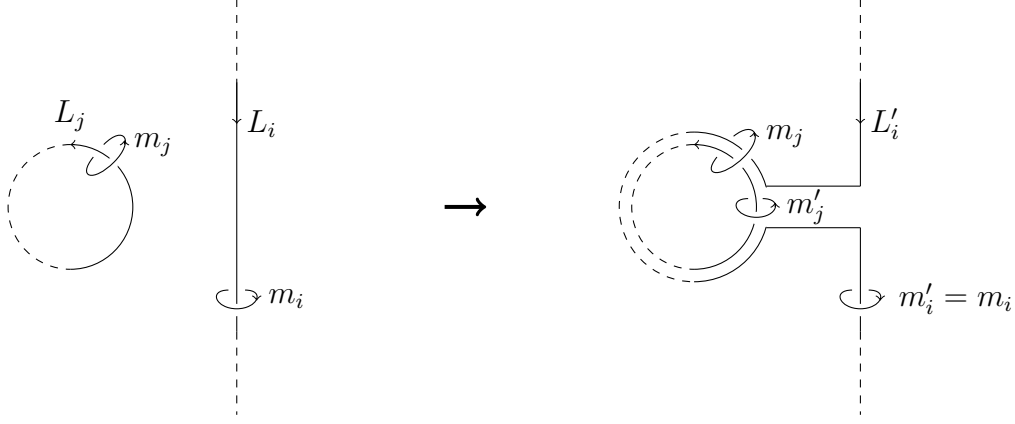


Figure 7: Effect of a K2 move on the meridians

presentation \mathbf{m}' of $H_1(M; \mathbb{Z})$ satisfies $P\mathbf{m}' = \mathbf{m}$.

Corollary 3.2. *Any 3-manifold admits an oriented surgery presentation whose associated linking matrix is admissible.*

4 Presentations of $H_1(M; \mathbb{Z})$, topological version

An *integral homology sphere*, or \mathbb{Z} -sphere, is a 3-manifold which has the same homology as the standard 3-sphere S^3 . Define oriented surgery presentations on links in \mathbb{Z} -spheres as in S^3 . The next proposition is an equivalent of Theorem 2.4 for surgery presentations in \mathbb{Z} -spheres. We will end the section by deducing Theorem 2.4 from this proposition and Matveev's result for \mathbb{Z} -spheres.

Proposition 4.1. *Let M be a 3-manifold. Let A be an admissible linking matrix. Assume there is an A -presentation $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)$ of $H_1(M; \mathbb{Z})$. Then there is a \mathbb{Z} -sphere N and an oriented surgery presentation $L \subset N$ of M such that $A(L) = A$ and L induces the A -presentation $\boldsymbol{\eta}$.*

Proof. The idea of this proof is to kill the homology of M by surgeries in order to obtain a \mathbb{Z} -sphere, and to consider the inverse surgeries.

Write $A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$ where $A_0 = (a_{ij})_{1 \leq i, j \leq m}$ is a non-degenerate linking matrix. For $1 \leq i \leq n$, let K_i be a framed knot in M whose homology class is η_i . Choose the K_i

pairwise disjoint. Let $T(K_i)$ be a tubular neighborhood of K_i , let $m(K_i) \subset \partial T(K_i)$ be an oriented meridian of K_i , and let $\ell(K_i) \subset \partial T(K_i)$ be the fixed parallel of K_i .

For $1 \leq i, j \leq m$, we have $lk(K_i, K_j) \equiv (A_0^{-1})_{ij} \pmod{\mathbb{Z}}$. Since adding to a K_i some meridians of the K_j does not modify the homology class of K_i in M (and since we can modify the choice of the $\ell(K_i)$), we may assume, for all $1 \leq i, j \leq m$, that:

$$lk(K_i, K_j) = (A_0^{-1})_{ij}.$$

Since the columns of A give relations on the η_i , there are 2-chains Σ_i in $X_0 = M \setminus \sqcup_{1 \leq i \leq m} \text{Int}(T(K_i))$ such that

$$\partial \Sigma_i = \sum_{j=1}^m a_{ij} \ell(K_j) + \sum_{j=1}^m b_{ij} m(K_j),$$

for some integers b_{ij} .

It follows from Poincaré duality that there are closed oriented surfaces S_i , for $m+1 \leq i \leq n$, such that

$$\langle S_i, \eta_j \rangle = \delta_{ij}$$

for $m+1 \leq i \leq n$ and $1 \leq j \leq n$, where δ_{ij} is the Kronecker symbol. Adding if necessary copies of the S_j to the Σ_i , and tubing around the $T(K_j)$, we can (and we do) assume that the Σ_i are embedded in $X = X_0 \setminus \sqcup_{m+1 \leq j \leq n} \text{Int}(T(K_j))$.

Let us compute the integers b_{ij} . For $1 \leq k \leq m$, let $\ell(K_k)_{ext}$ be a parallel copy of $\ell(K_k)$ in the interior of a regular neighborhood of $\partial T(K_k)$ in X . We have $\langle \Sigma_i, \ell(K_k)_{ext} \rangle = -b_{ik}$. On the other hand, this algebraic intersection number is equal to the linking number $lk(\partial \Sigma_i, \ell(K_k)_{ext}) = \sum_{j=1}^m a_{ij} lk(K_j, K_k) = \delta_{ik}$. Hence $b_{ik} = -\delta_{ik}$ and

$$\partial \Sigma_i = \sum_{j=1}^m a_{ij} \ell(K_j) - m(K_i).$$

Now let N be the 3-manifold obtained from M by surgery along the framed link $\sqcup_{1 \leq i \leq n} K_i$. The group $H_1(N; \mathbb{Z})$ is generated by the $[m(K_i)]$, which are easily seen to be trivial by considering the surfaces Σ_i and S_i . Hence N is an integral homology sphere. For $1 \leq i \leq n$, let $\hat{K}_i \subset N$ be the core of the torus $T(\hat{K}_i)$ reglued during the surgery. Orient each \hat{K}_i so that it is homologous to $-m(K_i)$ in $T(\hat{K}_i)$, and parallelize \hat{K}_i with the fixed parallel $\ell(\hat{K}_i) = -m(K_i)$. Note that $\ell(K_i)$ is an oriented meridian of \hat{K}_i . The manifold M is obtained from N by surgery along the framed link $L = \sqcup_{1 \leq i \leq n} \hat{K}_i$, and the associated generators of $H_1(M; \mathbb{Z})$ are the $[\ell(K_i)] = \eta_i$.

Let us compute the linking matrix $A(L)$. For $1 \leq i \leq m$, construct from Σ_i another 2-chain $\hat{\Sigma}_i$ by adding a_{ij} meridian disks in $T(\hat{K}_j)$ for $1 \leq j \leq m$, and an annulus in $T(\hat{K}_i)$, so that $\partial \hat{\Sigma}_i = \hat{K}_i$. For $1 \leq i, j \leq m$, we have $lk(\hat{K}_i, \hat{K}_j) = \langle \hat{\Sigma}_i, \ell(\hat{K}_j) \rangle = -a_{ij}$. Similarly, we have $lk(\hat{K}_i, \hat{K}_j) = 0$ when $1 \leq i \leq m$ and $m+1 \leq j \leq n$. For $m+1 \leq i \leq n$, $m(K_i)$ bounds the surface $-S_i \cap X$ which does not meet the \hat{K}_j , hence $lk(\hat{K}_i, \hat{K}_j) = 0$ for all $1 \leq j \leq n$. Finally, $A(L) = A$. \square

First proof of Theorem 2.4. Let N and $\hat{L} \subset N$ by the \mathbb{Z} -sphere and the oriented surgery presentation provided by Proposition 4.1. By [Mat87, Theorem 2], there is a Y-link $\Gamma \subset S^3$ such that $N \cong S^3(\Gamma)$. The genus 3 handlebodies in N reglued during the surgery can be viewed as regular neighborhoods of Y-graphs, hence we can assume that the link \hat{L} does not meet them, and we can consider the copy of \hat{L} in S^3 . Write Γ as the disjoint union of Y-graphs Γ_i for $1 \leq i \leq s$, and let $L \subset S^3$ be the framed link defined by the union of all the six-component links associated with the Γ_i 's and of \hat{L} . Number and orient the

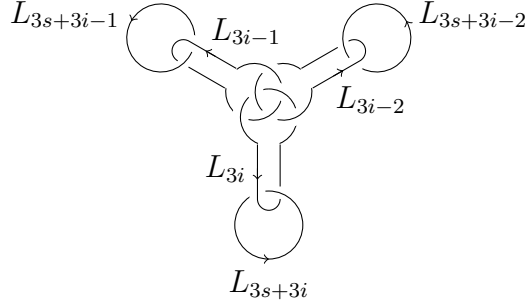


Figure 8: Six-component link associated with Γ_i

components of L as indicated in Figure 8 for the $6s$ first components, and set $L_{6s+i} = \hat{L}_i$. The linking matrix of L is:

$$A(L) = \begin{pmatrix} 0 & I_{3s} & 0 \\ I_{3s} & \star & C \\ 0 & {}^t C & A \end{pmatrix},$$

for some matrix C of size $3s \times n$. The associated presentation of $H_1(M; \mathbb{Z})$ is $(\boldsymbol{\gamma}, 0, \dots, 0, \boldsymbol{\eta})$, where $\boldsymbol{\gamma} = -C\boldsymbol{\eta}$. Define a congruence matrix $P = \begin{pmatrix} I_{3s} & 0 & -C \\ 0 & I_{3s} & 0 \\ 0 & 0 & I_n \end{pmatrix}$. Note that P is unimodular. By Lemma 3.1, L can be modified by Kirby moves to obtain an oriented

surgery presentation L' , with linking matrix $A(L') = {}^tPA(L)P = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}$ for some linking matrix B of determinant -1 , and whose associated presentation of $H_1(M; \mathbb{Z})$ is $\boldsymbol{\eta}' = (0, \dots, 0, \boldsymbol{\eta})$.

Performing if necessary a stabilization, we can assume that B is the matrix of an indefinite odd unimodular bilinear symmetric form over some power of \mathbb{Z} , and it follows that it is congruent to a diagonal matrix with ± 1 's on the diagonal (see for instance [MH73, Chap. 2, Th. 4.3]). Applying once again Lemma 3.1, we obtain an oriented surgery presentation L'' whose linking matrix is a stabilization of A , and whose associated presentation of $H_1(M; \mathbb{Z})$ is still $\boldsymbol{\eta}'$. \square

5 Presentations of $H_1(M; \mathbb{Z})$, algebraic version

In this section, we give an alternative proof of Theorem 2.4, more technical than the previous one, but also more constructive, in the sense that if two 3-manifolds M and N are given by surgery presentations in S^3 , and if an isomorphism $\xi : H_1(M; \mathbb{Z}) \cong H_1(N; \mathbb{Z})$ preserving the linking pairing is fixed, then the following proof provides Kirby moves which lead to the situation of Proposition 2.3, from which one can write down the Y-link realizing ξ .

Lemma 3.1 implies that Theorem 2.4 can be proved by showing that any two A -presentations of a given linked group are related, up to stabilizations, by a congruence which induces the required change of generators. We first treat the case of a finite linked group. The following proposition is a refinement of [Dur77, Theorem 4.1].

Proposition 5.1. *Let H be a finite linked group. Let A and B be non-degenerate linking matrices. Assume there are an A -presentation $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ and a B -presentation $\boldsymbol{\eta} = (\eta_1, \dots, \eta_m)$ of H . Then there are stabilizations \tilde{A} of A and \tilde{B} of B , and a unimodular matrix P , such that $\tilde{A} = {}^tP\tilde{B}P$ and $P\boldsymbol{\gamma}' = \boldsymbol{\eta}'$, where $\boldsymbol{\gamma}' = (0, \dots, 0, \gamma_1, \dots, \gamma_n)$ and $\boldsymbol{\eta}' = (0, \dots, 0, \eta_1, \dots, \eta_m)$.*

Proof. We first fix some formalism. A *linked lattice* is a finitely generated free \mathbb{Z} -module R equipped with a non-degenerate symmetric bilinear form $\psi_R : R \times R \rightarrow \mathbb{Q}$. It is *integral* if ψ_R takes values in \mathbb{Z} . Let V be a finite dimensional \mathbb{Q} -vector space equipped with a non-degenerate symmetric bilinear form $\psi : V \times V \rightarrow \mathbb{Q}$. A *lattice in V* is a free \mathbb{Z} -submodule R of maximal rank. It is linked by the form ψ . Define the dual module of R by:

$$R^\# = \{x \in V \text{ such that } \psi(x, y) \in \mathbb{Z} \text{ for all } y \in R\}.$$

The group R^\sharp naturally identifies with $\text{hom}(R; \mathbb{Z})$ via $x \mapsto (y \mapsto \psi(x, y))$. Note that $(R^\sharp)^\sharp = R$. The lattice R is *unimodular* if $R^\sharp = R$. If the linked lattice R is integral, then $R \subset R^\sharp$, and ψ induces a linking pairing on R^\sharp/R .

Set $V_1 = \mathbb{Q}^n$ and denote by $\mathbf{e} = (e_1, \dots, e_n)$ its canonical basis. Equip V_1 with the symmetric bilinear form ψ_A given by the matrix A in the basis \mathbf{e} . Set $R_1 = \mathbb{Z}^n \subset \mathbb{Q}^n$. Let $\mathbf{g} = (g_1, \dots, g_n)$ be the basis of the dual lattice R_1^\sharp dual to \mathbf{e} . Note that g_i is given in the basis \mathbf{e} of V_1 by the i -th column of A^{-1} . We have an identification of linked groups $H \cong R_1^\sharp/R_1$ given by $\gamma_i \mapsto \bar{g}_i$, where \bar{g}_i is the class of g_i modulo R_1 .

Similarly, using the matrix B , define a \mathbb{Q} -vector space $V_2 = \mathbb{Q}^m$ with canonical basis $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m)$, a symmetric bilinear form ψ_B on V_2 , a linked lattice $R_2 = \mathbb{Z}^m \subset V_2$, a dual lattice R_2^\sharp with a basis $\mathbf{h} = (h_1, \dots, h_m)$ dual to $\boldsymbol{\varepsilon}$, and an identification of the linked groups H and R_2^\sharp/R_2 identifying η_i with \bar{h}_i .

In H , we have $\gamma_i = \sum_{j=1}^m u_{ij} \eta_j$ and $\eta_i = \sum_{j=1}^n v_{ij} \gamma_j$, for some integers u_{ij} and v_{ij} . Define $f : R_1^\sharp \rightarrow R_2^\sharp$ by $g_i \mapsto \sum_{j=1}^m u_{ij} h_j$. The map f induces an isomorphism

$$\bar{f} : R_1^\sharp/R_1 \xrightarrow{\cong} R_2^\sharp/R_2$$

which is the identity on H via the given identifications.

Set $R = R_1 \oplus R_1^\sharp$. Define a linked lattice structure on R by:

$$\psi_R((x, y), (x', y')) = \psi_A(x, y') + \psi_A(x', y) + \psi_A(y, y') - \psi_B(f(y), f(y')).$$

The form ψ_R takes values in \mathbb{Z} since \bar{f} is an isomorphism of linked groups. The matrix of ψ_R in the basis (\mathbf{e}, \mathbf{g}) is of the form $\begin{pmatrix} 0 & I \\ I & * \end{pmatrix}$. Hence R is a unimodular linked lattice.

Consider the linked lattice $R \oplus R_2^\sharp$ equipped with the form $\Psi = \psi_R \oplus \psi_B$. Define $\iota : R_1^\sharp \rightarrow R \oplus R_2^\sharp$ by $y \mapsto (0, y, f(y))$. The map ι is injective and respects the bilinear forms. Set $S = (\iota(R_1^\sharp))^\perp \subset R \oplus R_2^\sharp$. Let us check that $S \subset R \oplus R_2$. Let $(x, y, z) \in S$. For $r \in R_1^\sharp$, $\Psi((x, y, z), (0, r, f(r))) = 0$ implies $\psi_B(z, f(r)) \in \mathbb{Z}$. Now each class of R_2^\sharp modulo R_2 contains an element $f(r)$, hence $\psi_B(z, s) \in \mathbb{Z}$ for all $s \in R_2^\sharp$, i.e. $z \in R_2$. Hence S equipped with the restriction of Ψ is an integral linked lattice. Define a map $\omega : S \oplus R_1^\sharp \rightarrow R \oplus R_2^\sharp$ as the direct sum of the inclusion $S \hookrightarrow R \oplus R_2^\sharp$ and of ι .

Lemma 5.2.

- The linked lattice S is unimodular.
- The map $\omega : S \oplus R_1^\sharp \rightarrow R \oplus R_2^\sharp$ is an isomorphism which respects the bilinear forms and identifies $S \oplus R_1$ with $R \oplus R_2$.

Proof. We first prove that ω is an isomorphism of linked lattices, *i.e.* that $R \oplus R_2^\sharp = S \oplus \iota(R_1^\sharp)$. Since A is non-degenerate and $S = (\iota(R_1^\sharp))^\perp$, we have $S \cap \iota(R_1^\sharp) = \{0\}$. Hence it suffices to prove $R \oplus R_2^\sharp = S + \iota(R_1^\sharp)$.

We have $R \oplus R_2 = S + R_1 \subset R \oplus R_2^\sharp$, where R_1 is the first component of $R = R_1 \oplus R_1^\sharp$. Indeed, for $(x, y, z) \in R \oplus R_2 = R_1 \oplus R_1^\sharp \oplus R_2$, define $\rho : R_1^\sharp \rightarrow \mathbb{Z}$ by $u \mapsto \psi_A(y, u) - \psi_B(f(y), f(u)) + \psi_B(z, f(u))$. Since $\rho \in \text{hom}(R_1^\sharp; \mathbb{Z})$, there is $w \in R_1$ such that $\rho(u) = -\psi_A(w, u)$. It follows that $(w, y, z) \in S$, thus $(x, y, z) \in S + R_1$.

We have $R \oplus R_2^\sharp = (R \oplus R_2) + \iota(R_1^\sharp)$. Indeed, if $y \in R_2^\sharp$, there is $y_0 \in R_2$ such that $y = y_0 + f(x)$ for some $x \in R_1^\sharp$. Hence $(0, 0, y) = (0, -x, y_0) + (0, x, f(x))$.

Finally $R \oplus R_2^\sharp = S + R_1 + \iota(R_1^\sharp)$. Now, for $x \in R_1$, $x - \iota(x) \in S$. Hence $R \oplus R_2^\sharp = S + \iota(R_1^\sharp) = S \oplus \iota(R_1^\sharp)$.

To show that S is unimodular, we prove that the map $S \rightarrow \text{hom}(S; \mathbb{Z})$ defined by $x \mapsto (y \mapsto \Psi(x, y))$ is bijective. Injectivity follows from $R \oplus R_2^\sharp = S \oplus^\perp \iota(R_1^\sharp)$. Let us check surjectivity. Let $\phi \in \text{hom}(S; \mathbb{Z})$. Extend it into $\tilde{\phi} \in \text{hom}(R \oplus R_2^\sharp; \mathbb{Z})$ by 0 on $\iota(R_1^\sharp)$. Then $\tilde{\phi} = \Psi(v, \cdot)$ for some $v \in (R \oplus R_2^\sharp)^\sharp = R \oplus R_2$. Since $\tilde{\phi}|_{\iota(R_1^\sharp)} = 0$, $v \in S$ and $\phi = \Psi(v, \cdot)$.

Finally, taking the dual lattices in the equality $R \oplus R_2^\sharp = S \oplus \iota(R_1^\sharp)$, we obtain $R \oplus R_2 = S \oplus \iota(R_1)$ since R and S are unimodular. \square

Back to the proof of the proposition, fix a basis \mathbf{s} of S . The form Ψ on $S \oplus R_1 \cong R \oplus R_2$ is given by the matrices

$$\hat{A} = \begin{pmatrix} C & 0 \\ 0 & A \end{pmatrix} \text{ in } (\mathbf{s}, \iota(\mathbf{e})) \text{ and } \hat{B} = \begin{pmatrix} D & 0 \\ 0 & B \end{pmatrix} \text{ in } ((\mathbf{e}, \mathbf{g}), \boldsymbol{\varepsilon}),$$

where C and D are unimodular matrices. We have ${}^t P \hat{B} P = \hat{A}$, where

$$P = \begin{pmatrix} * & 0 \\ * & A \\ * & F \end{pmatrix},$$

and F is the matrix of f in the bases \mathbf{e} and $\boldsymbol{\varepsilon}$. Let us prove that $P\boldsymbol{\gamma}' = \boldsymbol{\eta}'$.

Set $U = (u_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ and $V = (v_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$. The matrix of f in the bases \mathbf{g} and \mathbf{h} is ${}^t U$. Hence $F = B^{-1} {}^t U A$. Moreover, since f is an isomorphism of linked groups, we have $A^{-1} = U B^{-1} {}^t U \text{ mod } \mathcal{M}_n(\mathbb{Z})$.

For $1 \leq i \leq n$, $\eta_i = \sum_{k=1}^m (VU)_{ik} \eta_k$ in H . Hence $\sum_{k=1}^m (VU)_{ik} h_k - h_i \in R_2$. Thus

$VU = I \bmod \mathcal{M}_m(\mathbb{Z})B$. Finally:

$$\begin{aligned} V &= VUB^{-1}{}^tUA \bmod \mathcal{M}_{m,n}(\mathbb{Z})A \\ &= B^{-1}{}^tUA \bmod \mathcal{M}_{m,n}(\mathbb{Z})A \\ &= F \bmod \mathcal{M}_{m,n}(\mathbb{Z})A. \end{aligned}$$

Hence there is a matrix G such that $V = F + GA$, and

$$P\gamma' = \begin{pmatrix} * & 0 \\ * & A \\ * & V - GA \end{pmatrix} \begin{pmatrix} 0 \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ A\gamma \\ (V - GA)\gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix} \text{ in } H^{2n+m}.$$

It remains to modify \hat{A} and \hat{B} into stabilizations of A and B without acting on the presentations γ' and η' .

For a unimodular linked lattice Z with form ψ_Z , it is known that if ψ_Z is indefinite (there is $z \in Z$ such that $\psi_Z(z, z) = 0$) and odd (there is $z \in Z$ such that $\psi_Z(z, z)$ is odd), then there is a basis of Z in which the matrix of ψ_Z is diagonal (see for instance [MH73, Chap. 2, Th. 4.3]). Making if necessary a direct sum of R and S with a copy of \mathbb{Z} equipped with a form given by 1 or -1 on a generator, we can assume that R and S are equipped with an odd indefinite form. Hence the matrices C and D are congruent to diagonal matrices with ± 1 on the diagonal. Applying these congruences on \hat{A} and \hat{B} , we obtain stabilizations \tilde{A} and \tilde{B} of A and B related by a congruence such that the congruence matrix acts on γ' as P . \square

Second proof of Theorem 2.4. Thanks to Lemma 3.1, and recalling that any linking matrix is congruent to an admissible one, we may assume that the linking matrix A is admissible.

Write $A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$, where B is a non-degenerate linking matrix. Let L be an oriented

surgery presentation of M such that $A(L) = \begin{pmatrix} B_L & 0 \\ 0 & 0 \end{pmatrix}$ where B_L is a non-degenerate linking matrix. Let \mathbf{m} be the associated presentation of $H_1(M; \mathbb{Z})$. Let β be the rank of $H_1(M; \mathbb{Z})$, and let k (resp. ℓ) be the size of B (resp. B_L). We shall modify the presentation η so that the β last generators coincide with those of \mathbf{m} . There are matrices C and D , with integral coefficients, such that $\det(D) = \pm 1$, and

$$\begin{pmatrix} m_{\ell+1} \\ \vdots \\ m_{\ell+\beta} \end{pmatrix} = C \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_k \end{pmatrix} + D \begin{pmatrix} \eta_{k+1} \\ \vdots \\ \eta_{k+\beta} \end{pmatrix}.$$

Set $P = \begin{pmatrix} I & 0 \\ C & D \end{pmatrix}$. We have ${}^tPAP = A$ and $P\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_k \\ m_{\ell+1} \\ \vdots \\ m_{\ell+\beta} \end{pmatrix}$.

By Proposition 5.1, there are stabilizations \tilde{B} of B and \tilde{B}_L of B_L , and a unimodular matrix Q , such that ${}^tQ\tilde{B}_LQ = \tilde{B}$ and $Q \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \eta_1 \\ \vdots \\ \eta_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_1 \\ \vdots \\ m_\ell \end{pmatrix}$.

Set $\tilde{P} = \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix}$ (note that the size of the blocs is different for the two matrices). Set $\tilde{A}(\tilde{L}) = \begin{pmatrix} \tilde{B}_L & 0 \\ 0 & 0 \end{pmatrix}$. Then $\tilde{A} = {}^t\tilde{P}\tilde{A}(\tilde{L})\tilde{P}$ is a stabilization of A and $\boldsymbol{\eta}' = (0, \dots, 0, \eta_1, \dots, \eta_{k+\beta})$ is an \tilde{A} -presentation of $H_1(M; \mathbb{Z})$ such that $\tilde{P}\boldsymbol{\eta}' = \boldsymbol{m}' = (0, \dots, 0, m_1, \dots, m_{\ell+\beta})$. Up to stabilization, we can assume that $\det(\tilde{P}) = 1$.

Perform stabilizations of L in order to obtain a surgery link with linking matrix $\tilde{A}(\tilde{L})$. Then apply Lemma 3.1 to obtain a surgery link with linking matrix \tilde{A} and associated presentation $\boldsymbol{\eta}'$ of $H_1(M; \mathbb{Z})$. \square

6 Example

We give in this section an example of a borromean surgery which realizes a non-trivial isomorphism of the first homology group of a given 3-manifold, namely the multiplication by -1 . The method can be applied to any surgery presentation; we consider here a presentation given by a non-invertible knot in order to make things more explicit.

The knot 9_{32} drawn in Figure 9 is a non-invertible knot, *i.e.* the two possible orientations provide non-isotopic oriented knots. Let 9_{32}^+ be the one represented in Figure 9, and let 9_{32}^- be the other one. Let $K \subset S^3$ be the knot 9_{32}^+ with framing $n > 1$. Any knot can be obtained from the unknot with the same framing by borromean surgeries, and conversely (see the proof of Proposition 2.3). Figure 9 shows the knot K as the connected sum of an unknot and a 9_{32}^+ . The borromean surgery on the drawn Y-link Γ modifies the right part

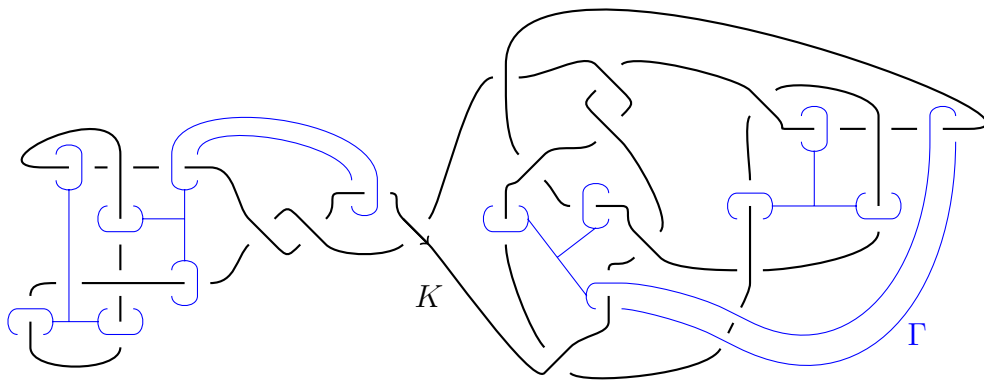


Figure 9: A Y-link inverting the knot 9_{32}

by trivializing the 9_{32}^+ , and the left part by transforming the unknot into a 9_{32}^- . Hence it globally changes the 9_{32}^+ into a 9_{32}^- .

Set $M = S^3(K)$. Note that $H_1(M; \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ is non-trivial and finite. The manifolds M and $M(\Gamma)$ are both obtained from S^3 by surgery on a knot 9_{32} , hence they are homeomorphic. However, the oriented meridians of the oriented surgery presentations given by 9_{32}^+ and 9_{32}^- have opposite homology classes. Hence the Y-link Γ does not modify the manifold M , up to homeomorphism, but it realizes the non-trivial isomorphism of $H_1(M; \mathbb{Z})$ given by multiplication by -1 .

References

- [CGP14] F. COSTANTINO, N. GEER & B. PATUREAU-MIRAND – “Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories”, *Journal of Topology* **7** (2014), no. 4, p. 1005–1053.
- [Dur77] A. H. DURFEE – “Bilinear and quadratic forms on torsion modules”, *Advances in Mathematics* **25** (1977), no. 2, p. 133–164.
- [GGP01] S. GAROUFALIDIS, M. GOUSSAROV & M. POLYAK – “Calculus of clovers and finite type invariants of 3-manifolds”, *Geometry & Topology* **5** (2001), p. 75–108.

- [Gou99] M. GOUSSAROV – “Finite type invariants and n -equivalence of 3-manifolds”, *Comptes Rendus de l’Académie des Sciences, Série I, Mathématiques* **329** (1999), no. 6, p. 517–522.
- [Hab00] K. HABIRO – “Claspers and finite type invariants of links”, *Geometry and Topology* **4** (2000), p. 1–83.
- [Kir78] R. KIRBY – “A calculus for framed links in S^3 ”, *Inventiones Mathematicae* **45** (1978), no. 1, p. 35–56.
- [KP53] M. KNESER & D. PUPPE – “Quadratische formen und verschlingungsinvarianten von knoten”, *Mathematische Zeitschrift* **58** (1953), no. 1, p. 376–384.
- [Kyl54] R. KYLE – “Branched covering spaces and the quadratic forms of links”, *Annals of Mathematics* (1954), p. 539–548.
- [Lic62] W. R. LICKORISH – “A representation of orientable combinatorial three-manifolds”, *Annals of Mathematics* (1962), p. 531–540.
- [Mas03] G. MASSUYEAU – “Spin Borromean surgeries”, *Transactions of the American Mathematical Society* **355** (2003), no. 10, p. 3991–4017.
- [Mas10] — , “Some finiteness properties for the Reidemeister–Turaev torsion of three-manifolds”, *Journal of Knot Theory and Its Ramifications* **19** (2010), no. 3, p. 405–447.
- [Mat87] S. V. MATVEEV – “Generalized surgeries of three-dimensional manifolds and representations of homology spheres”, *Matematicheskie Zametki* **42** (1987), no. 2, p. 268–278.
- [MH73] J. W. MILNOR & D. HUSEMÖLLER – *Symmetric bilinear forms*, Springer, 1973.
- [MN89] H. MURAKAMI & Y. NAKANISHI – “On a certain move generating link-homology”, *Mathematische Annalen* **284** (1989), no. 1, p. 75–89.
- [Wal60] A. H. WALLACE – “Modifications and cobounding manifolds”, *Canadian Journal of Mathematics* **12** (1960), p. 503–528.